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The propagation of a weak nonlinear wave

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We consider the propagation of a weak nonlinear wave whose energy is concentrated in a narrow band of wavenumbers in a fluid which is both dispersive and dissipative. We use the small amplitude equations of Whitham's theory of slowly varying wave trains, modified slightly to include dissipation, to show that the modulation of the wave may be described by a nonlinear Schrödinger equation. For long waves which are purely dispersive we obtain the Korteweg–de Vries equation, and for long waves which are dissipative we obtain Burgers' equation by suitable transformations of the nonlinear Schrödinger equation. We mention the problem of Stokes waves in deep water and comment briefly upon invariant far-field theory.

1. Introduction

Taniuti & Washimi (1968) considered the modulational instability of a small, but finite amplitude, dispersive hydromagnetic wave propagating in a cold quasi-neutral plasma. They found that in a frame of reference moving downstream with the group velocity the slow variation in the complex amplitude A of the wave could be described by a nonlinear Schrödinger equation of the form

$$i\frac{\partial A}{\partial t}+p\frac{\partial^2 A}{\partial x^2} = iqA+r|A|^2A, \tag{1}$$

where $|A|$ denotes the modulus of A. In (1) the space co-ordinate x is in the direction of propagation, t represents the time, and p, q and r are (in general) determinable constants. In their problem there was no dissipation and in consequence $q = 0$ and p and r are real. To obtain (1), which is their far-field equation, they used the method of multiple scales. Another paper which considered a similar kind of problem is that by Watanabe (1969).

Stewartson & Stuart (1971) have also used the method of multiple scales to study the growth of a small disturbance in plane Poiseuille flow when the Reynolds number is somewhat larger than its critical value. Their wave is not only dispersive but also dissipative, and they found that the modulation of the wave could also be described by (1) but with p and r as complex constants and q real but non-zero. Also, DiPrima, Eckhaus & Segel (1971) were able to show by using a discrete modal analysis that (1) describes the growth of a small disturbance in the vicinity of a marginally stable state for a fairly wide class of fluid-dynamical problems.

The method of multiple scales can often be long and intricate and our main aim in this paper is to present a simple derivation, valid for a rather general weak nonlinear wave, of the Schrödinger equation which describes the modulation of the wave amplitude in the far field. In § 2 we shall show that, except for special cases, the nonlinear Schrödinger equation may be derived directly from the averaging technique developed by Whitham (1965, 1967) and Lighthill (1965), modified to allow for dissipation. We shall also indicate how generalizations can be written down with the minimum of trouble. The method which we shall use bears some resemblance to that of Benney & Newell (1967); also, as we do not use the Navier–Stokes equations it is of considerable interest that we obtain the same equation as that found by Stewartson & Stuart (1971), who did start with the Navier–Stokes equations. Indeed, although our principal concern in this paper is with waves in fluids, much of what we have to say is also relevant to waves in solids and lattices (e.g. see Tappert & Varma 1970).

In § 3 we propound the importance of the general case and the key role of the nonlinear Schrödinger equation in this subject, and we give reasons as to why we feel that in the past special cases have received undue attention. For instance, we demonstrate that both the Korteweg–de Vries equation and Burgers' equation may be derived from the corresponding Schrödinger equations for waves which are 'not quite so long'. Another interesting fact which we consider in this section is that the steady-state solutions of the Korteweg–de Vries equation and of the Burgers equation comprise *exactly* the steady-state solutions of the Schrödinger equation.

In § 4 we show, as a simple example, that the equations which were used by Chu & Mei (1971) to describe the motion of Stokes waves in deep water may be rewritten as the appropriate Schrödinger equation. We also discuss a few problems from the point of view of invariant far-field theory and the method of multiple scales. It is worth noting that, whereas in general the propagation of a wave's modulation is governed by a Schrödinger equation, for long waves and for perturbations about equilibrium states (when the wave amplitude is nearly constant), a further reduction may often be made to either the Korteweg–de Vries equation or to the Burgers equation.

2. Derivation of the Schrödinger equation

Stewartson & Stuart (1971) considered the nonlinear growth of an initially infinitesimally small disturbance in plane Poiseuille flow for a Reynolds number slightly larger than the critical value. In such a problem the energy of the disturbance is concentrated in a narrow band of wavenumbers. This is the kind of problem in which we are interested, and we mention this now because it may help the reader to understand the ideas behind our analysis. We suppose that the disturbance is superimposed on a basic steady flow which may be characterized by non-dimensional parameters such as the Reynolds number and the Prandtl number and that the physical properties of the fluid remain constant. Thus we suppose that there exist a reference length L, a reference velocity U and a reference time L/U with respect to which all quantities are made non-dimensional.

Let the disturbance propagate in an unbounded positive x direction so that it may be described by a total wave function $\hat{\psi} = B\exp\{i(kx-\omega t)\}$, where B, k and ω are slowly varying functions of x and the time t. We restrict the wavenumber k to real values; here B is a pseudo-amplitude only, the true amplitude being $A = B\exp\{\omega_i t\}$, where $\omega = \omega_r + i\omega_i$ and the suffixes r and i denote real and imaginary parts respectively. When the wave is purely dispersive $\omega_i = 0$ and there is no distinction between A and B. The full equations and boundary conditions of the specific problem under consideration will require ω and k to be related, when the wave is infinitesimally small and unmodulated, by a functional condition:

$$D(k,\omega) = 0. \tag{2}$$

For non-dissipative problems (2) is usually called the linear dispersion relationship.

It follows from (2) that if for the linearized problem a solution Γ of the form $\exp\{i(kx-\omega t)\}$ requires $D(k,\omega) = 0$, that is $D(-i\partial/\partial x, i\partial/\partial t)\,\Gamma = 0$, then if we seek a solution of the form $\Gamma = B(x,t)\exp\{i(kx-\omega t)\}$ the associated variational equation of Whitham's theory may be written as

$$D(k,\omega)\,B - \tfrac{1}{2}\{D_{kk}B_{xx} - 2D_{k\omega}B_{xt} + D_{\omega\omega}B_{tt}\} + \tfrac{1}{6}i\{D_{kkk}B_{xxx} + \text{similar higher order terms}\}\dagger + \text{nonlinear term} = 0, \tag{3}$$

where suffixes denote partial derivatives. A term $-i\{D_k B_x - D_\omega B_t\}$ has been omitted from (3) because it balances the dominant terms due to variations of k and ω with respect to x and t. This balance is expressed by the energy-transport equation, which we shall use separately later.

For a real physical problem the boundary conditions will usually require $D(k,\omega)$ to be a complicated function, and except for special cases we shall neglect the higher order terms in (3) and truncate at quadratic order in derivatives of both k and ω. This process implies the invocation of a far-field approximation, or, to put it another way, that the energy of the wave group is concentrated in the neighbourhood of a point (k_0, ω_0) in the phase space. In general ω will be an analytic function of k, or *vice versa*, 'almost everywhere' in the phase space. Special cases when this is not so will need separate treatment (for an example see Midzuno & Watanabe 1970) but we shall suppose that (2) may be written as $\omega = \Omega(k)$ near some point (k_0, ω_0) in the phase space. When $k = k_0$ we shall write $\Omega(k_0) = \Omega_0 = \Omega_{0r} + i\Omega_{0i}$, where the suffix 0 denotes the original state, as determined by some suitable initial condition. Unless otherwise stated we shall choose k_0 so as to make Ω_{0i} a *maximum* because if the growth of a small disturbance is controlled initially by linearized theory then the energy will collect into a neighbourhood of that wavenumber which makes the imaginary part of Ω a maximum.

So we put $D(k,\omega) \equiv \omega - \Omega(k)$ in (3) and neglect the higher order terms to obtain

$$\{\omega - \Omega(k)\}\,B - \tfrac{1}{2}\{-\Omega''(k)\,B_{xx}\} + \text{nonlinear term} = 0, \tag{4}$$

where a prime denotes differentiation with respect to k. In Whitham's theory the nonlinear term is replaced by the average value of the slowly varying part

† We note that for Klein–Gordon problems the higher order terms are identically zero.

over one cycle multiplied by the rapidly varying part. So in (4) we may suppose that the nonlinear term contributes to ω a function of A^2. Thus for a small but finite wave we may expand this function for small values of A^2, neglecting terms of order A^4, and rewrite (4) as

$$\omega = \Omega(k) - \frac{1}{2}\left(\frac{\partial^2\Omega}{\partial k^2}\right)_0 \frac{B_{xx}}{B} + \alpha A^2, \tag{5}$$

where we define

$$\alpha \equiv (\partial\omega/\partial A^2)_0. \tag{6}$$

The second term on the right-hand side of (5) was also obtained by Chu & Mei (1970); the coefficient of B_{xx}/B may be evaluated at $k = k_0$ as the error involved is of the same order as that of the higher order derivatives of B neglected.

We are interested in the departure of the quantity $kx - \omega t$ from its original value, so we follow Lighthill (1965) and define

$$\theta \equiv kx - \omega t = k_0 x - \Omega_0 t + \phi(x, t), \tag{7}$$

so that

$$\omega(x, t) = \Omega_0 - \phi_t \tag{8}$$

and

$$k(x, t) = k_0 + \phi_x. \tag{9}$$

We substitute (8) and (9) in (5) and ignore terms of order ϕ_x^3 in the expansion of $\Omega(k)$ to obtain

$$\Omega_0 - \phi_t = \Omega_0 + \left(\frac{\partial\Omega}{\partial k}\right)_0 \phi_x + \frac{1}{2}\left(\frac{\partial^2\Omega}{\partial k^2}\right)_0 \left\{\phi_x^2 - \frac{B_{xx}}{B}\right\} + \alpha A^2. \tag{10}$$

Next we define β and γ by

$$\beta \equiv (\partial\Omega/\partial k)_0, \quad \gamma \equiv (\partial^2\Omega/\partial k^2)_0, \tag{11}$$

so that we may rewrite (10) as

$$\phi_t + \beta\phi_x + \tfrac{1}{2}\gamma(\phi_x^2 - [B_{xx}/B]) + \alpha A^2 = 0. \tag{12}$$

Now (12) is a single equation which connects ϕ and B (for $A = B\,e^{\omega_i t}$) so we need another equation, the energy-transport equation, to complete the system. We shall use Whitham's energy-transport equation in its linear form only, although for stronger waves we would need to consider the dependence of the group velocity on the amplitude, as we shall indicate in § 4. For waves which are purely dispersive and have amplitude C, the linear energy-transport equation is

$$\frac{\partial C^2}{\partial t} + \frac{\partial}{\partial x}(c_g C^2) = 0, \tag{13}$$

where c_g is the infinitesimal group velocity. The amplitude $A = B\,e^{\omega_i t}$ of our wave, which in addition is dissipative, has this amplitude multiplied by the factor $e^{\omega_i t}$ because of dissipation, so that there exists an analogous wave of amplitude $C = A\,e^{-\omega_i t} = B$ which is purely dispersive, and therefore

$$\frac{\partial B^2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial k} B^2\right) = 0. \tag{14}$$

We note that in terms of A we may write (14) as

$$\frac{\partial A^2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial k} A^2\right) = 2\omega_i A^2, \tag{15}$$

if we use the property that ω_i is only a slowly varying function of x and t. In a private communication Professor Whitham has mentioned that one of his students (Jimenez) has also derived (15) by using some ideas of Prigogine. For small values of $k-k_0$, we have $\partial\Omega/\partial k = \beta+(k-k_0)\gamma = \beta+\gamma\phi_x$ from (9), so from (14) we obtain

$$\frac{\partial B^2}{\partial t}+\beta\frac{\partial B^2}{\partial x}+\gamma\frac{\partial}{\partial x}(\phi_x B^2) = 0. \tag{16}$$

This is the second equation which we required and, with (12), suffices to determine the system.

We now put $\chi = Be^{i\phi}$ and may thus readily verify that (12) and (16) are equivalent to the single equation

$$\frac{\partial \chi}{\partial t}+\beta\frac{\partial \chi}{\partial x}-\tfrac{1}{2}i\gamma\frac{\partial^2 \chi}{\partial x^2} = -i\alpha A^2\chi. \tag{17}$$

To describe the slowly varying part of the wave we define a wave function $\psi(x,t)$ by

$$\hat{\psi} \equiv \psi(x,t)\exp\{i(k_0x-\Omega_{0r}t)\}, \tag{18}$$

so that

$$\chi = \psi\exp(-\Omega_{0i}t) \tag{19}$$

and also $A = |\psi|$; we recall that $\hat{\psi}$ was defined earlier by $\hat{\psi} \equiv B\exp\{i(kx-\omega t)\}$. We now substitute for χ from (19) into (17) and find that the required equation for ψ is

$$\frac{\partial \psi}{\partial t}+\beta\frac{\partial \psi}{\partial x}-\tfrac{1}{2}i\gamma\frac{\partial^2 \psi}{\partial x^2} = \Omega_{0i}\psi-i\alpha|\psi|^2\psi. \tag{20}$$

We see from (20) that the appropriate time scale for the variation of ψ is $\tau = \Omega_{0i}t$. As ψ describes the slowly varying modulation of the wave we require Ω_{0i} to be small. As mentioned earlier, we may suppose, for example, that an initially infinitesimal disturbance is growing slowly, so that Ω_{0i} will indeed be small. For in many physical problems a given basic flow will become unstable to infinitesimal disturbances when a non-dimensional parameter P exceeds some critical value P_c. Then, in the k, P plane, the line $P = P_c$ is a tangent to the neutral stability curve $\Omega_{0i} = 0$ and touches this curve at what we shall call the *critical point*.

When P only exceeds P_c by a small amount the bandwidth $k-k_0$ of the energy-containing wavenumbers will be small, equal to ϵ say, and both Ω_{0i} and $P-P_c$ will be of order ϵ^2 provided that $\gamma \neq 0$. So we may expand α, β and γ in (20) in terms of ϵ and, consistently with those terms which have already been neglected, evaluate them at the critical point. At this point β is *real* and equal to the infinitesimal group velocity. In fact β will be real even for $P > P_c$ because, when the wave is infinitesimally small, linearized theory will select k_0 so that the imaginary part of Ω is a maximum, as mentioned earlier. So we now move the x axis downstream at the group velocity β and write (20) as

$$i\frac{\partial \psi}{\partial t}+\tfrac{1}{2}\gamma\frac{\partial^2 \psi}{\partial x^2} = i\Omega_{0i}\psi+\alpha|\psi|^2\psi. \tag{21}$$

Equation (21) is the principal equation of this paper; it is a dissipative form of the nonlinear Schrödinger equation, not only because of the term containing Ω_{0i}, but also as in general both α and γ will be complex constants. The imaginary part of γ will be negative for k_0 is such that the imaginary part of Ω is a maximum. The contribution $\alpha|\psi|^2$ to the nonlinear potential $V \equiv i\Omega_{0i}+\alpha|\psi|^2$ is due partly to the back-reaction of the distorted basic flow on the disturbance. In this context we can recognize an analogy with the problem of wave propagation in superfluid helium as studied by Tsuzuki (1971).

If it had been appropriate to use (2) in the form $k = \Omega^{-1}(\omega)$ rather than $\omega = \Omega(k)$ we would have obtained an equation similar to (21) but with the roles of x and t interchanged, as in the plasma problem considered by Watanabe (1969). It is crucial to move with the group velocity so as to remain near the centre of the wave group, for this is where the most important changes will take place, as Benney & Newell (1967) have observed. Details of the flow near the centre of the wave group may then be sought without obtaining a global solution. For instance the solution of a linear Klein–Gordon problem in such a neighbourhood may be expressed in the form

$$\psi = \sum_{i=1}^{\infty} \epsilon^i V_i\{\epsilon(x-c_g t), \epsilon^2 t\}, \tag{22}$$

where each V_i satisfies a Schrödinger equation.

We can see quite clearly where the Schrödinger operator comes from by considering an integral representation of the *linear* problem.† Let the energy of the disturbance be concentrated in a small wavenumber bandwidth near $k = k_0$. Then, as the problem is linear the total wavefunction Λ may be represented by an integral of the form

$$\Lambda = \int F(k-k_0)\, e^{i(kx-\omega t)}\, dk. \tag{23}$$

We put $k = k_0+\kappa$, so that κ is of order ϵ, and expand $\omega(k_0+\kappa)$ for κ small as $\omega_0+\kappa\omega_0'+\frac{1}{2}\kappa^2\omega_0''+\ldots$ so that we may write (23) as

$$\Lambda = e^{i(k_0x-\omega_{0r}t)} \int F(\kappa) \exp[\omega_{0i}t] \exp[i\{\kappa x-\kappa\omega_0' t-\tfrac{1}{2}\kappa^2\omega_0'' t\}]\, d\kappa. \tag{24}$$

We choose k_0 so that the imaginary part of ω is a maximum, ω_0' then being real, and move the x axis downstream at this group velocity. Now let ψ represent the slowly varying part of Λ, so that

$$\psi = \int F(\kappa) \exp[\omega_{0i}t] \exp[i\{\kappa x-\tfrac{1}{2}\kappa^2\omega_0'' t\}]\, d\kappa. \tag{25}$$

Then, by straightforward differentiation, it follows from (25) that

$$i\frac{\partial\psi}{\partial t}+\tfrac{1}{2}\omega_0''\frac{\partial^2\psi}{\partial x^2} = i\omega_{0i}\psi. \tag{26}$$

The values of ω_0' and ω_0'' used above in the expansion of ω may be identified with the values of β and γ as defined by (11). It is now easy to see that if $\gamma = 0$ in (21) then a higher x derivative term will be present, a situation which we will

† I am very grateful to Professor N. C. Freeman for this suggestion.

discuss in § 3. The above argument may be compared with the discussion in § 2 of Hocking, Stewartson & Stuart (1972).

For a given specific problem the values of α and γ in (21), and of β, must be found from the full equations and boundary conditions. If the flow is bounded in a z direction normal to the x axis then β and γ are determined by the eigenvalues of the equation for the z-dependent eigenfunction of linearized theory when the disturbance is infinitesimally small. The value of α, which is often called the first Landau constant, is determined by a secularity condition as is well explained, for example, by Reynolds & Potter (1967) and by Matkowsky (1970).

If the flow is also unbounded in a y direction, so that x, y and z form a right-handed system, we may consider the wavenumber k to be a vector in the x, y plane with a component l in the y direction. Then if $\mathbf{i}$ and $\mathbf{j}$ denote unit vectors in the x and y directions respectively we may generalize (9) to $\mathbf{k} = (k_0 + \phi_x)\,\mathbf{i} + (l_0 + \phi_y)\,\mathbf{j}$ and show, though the algebra is more cumbersome, that an additional term $\frac{1}{2}(\partial^2\Omega/\partial l^2)_0\,\partial^2\psi/\partial y^2$ must be added to the left-hand side of (21). For stability problems to which Squire's (1933) theorem is applicable the coefficient of $\partial^2\psi/\partial y^2$ will be evaluated at $l_0 = 0$, and a careful examination of (2), which will now be of the form $D(k, l, \omega) = 0$, will enable us to express this coefficient in terms of already known quantities. In a purely dispersive problem, for instance, its value is $\mu/2k_0$, where μ is the group velocity minus the phase velocity. In this case the coefficients of both $\partial^2\psi/\partial x^2$ and $\partial^2\psi/\partial y^2$ will be real and so, by appropriate scaling, these two terms may be written as $\nabla^2\psi$, where ∇^2 is the two-dimensional Laplacian operator.

In the derivation of (21) we have neglected higher order nonlinear terms on the right-hand side such as $\frac{1}{2}(\partial^2\omega/\partial(A^2)^2)_0\,|\psi|^4\,\psi$. If, however, we expand ψ in powers of the small parameter ϵ so that the leading term in this expansion is $\epsilon\psi_1$, then the term given above does not contribute to the equation for ψ_1 when ϵx and $\epsilon^2 t$ are of order one. Each term in (21) is strictly of order $\epsilon^3\psi_1$ or $\epsilon^3\psi_1^3$ and, at worst, we have neglected terms of order $\epsilon^5\psi_1^5$. Hence (21) is valid provided $\epsilon\psi_1 \ll 1$, and the theory breaks down when $\psi_1 \sim \epsilon^{-1}$. Thus a solution for ψ_1 found from (21) can be quite large and still consistent with the neglect of higher order nonlinear terms. Hocking & Stewartson (1972*a*, *b*) have recently undertaken a numerical and analytical study of the equation for ψ_1 including the y derivative term to determine whether, given initial conditions appropriate to the linearized problem, ψ_1 remains finite in size or bursts after a time of order ϵ^{-2}.

It is significant that the only simple term of order ϵ^3 which may be added to (21), without destroying the far-field invariance (see § 4), is the nonlinear term $\psi\psi_x$, which replaces $|\psi|^2\,\psi$ in long-wave theory as we shall discover in the next section. We defer until § 4 a discussion as to why (21) had to be a form of Schrödinger's equation, even if only in a disguised form.

3. The general case and the long-wave approximation

We mentioned in § 2 that we consider a problem such as the stability of plane Poiseuille flow to be a good example of the 'general case'. By the general case we mean one for which the analysis of § 2 is valid; in particular, this requires both k_0 and $(\partial^2\Omega/\partial k^2)_0$ to be non-zero. When one or both of these quantities is zero a special treatment is required and we have a 'special case'. For example, a stability problem will be a general case only if at the critical point on the neutral curve for infinitesimal disturbances k and $\partial^2\Omega/\partial k^2$ are both non-zero. It is not essential for the group velocity to be non-zero; if it is zero then typical problems are those of Bénard convection and of flow between rotating cylinders. Problems for which the group velocity is non-zero are, however, of more interest because then a disturbance will evolve over a long *distance*, so the modulation will be important provided that there is no recycling of the flow, as Stewartson & Stuart (1971) point out.

The general case is very important indeed and the role of the nonlinear Schrödinger equation in this subject has been obscured in the past because so much attention has been given instead to special cases, for which the relevant far-field equation is often either the Korteweg–de Vries equation or Burgers' equation. It is not sufficiently appreciated that these two equations are both intimately associated with the nonlinear Schrödinger equation as we shall see later in this section.

In the special case when $(\partial^2\Omega/\partial k^2)_0$ or γ in (21) is zero it is necessary to modify (5) to include a higher derivative of B with respect to x. Let γ_m be the mth derivative of Ω with respect to k at $k = k_0$, where m is the smallest integer $\geqslant 3$ such that $\gamma_m \neq 0$. Now, instead of (21), the method of § 2 yields the more general equation

$$i\frac{\partial\psi}{\partial t} - (-i)^m \frac{\gamma_m}{m!}\frac{\partial^m\psi}{\partial x^m} = i\Omega_{0i}\psi + \alpha|\psi|^2\psi, \tag{27}$$

where again the x co-ordinate moves downstream at the group velocity. To check that the linear terms in (27) are correct we may use the integral argument of § 2 by extending (24) to include the mth derivative term; the relevant scaled variables will now be κx and $\kappa^m t$, with κ of order ϵ and Ω_{0i} of order ϵ^m. For a problem with dissipation m will be an even integer, for if m were odd then the neutral curve for infinitesimal disturbances would contain another critical point at a smaller value of the physical parameter P. Thus, when m is an odd integer there will be no dissipation; this result has a connexion with the theory of long dispersive waves, for which (2) can be written as $i\omega = f(ik)$, where f is a real operator and an *odd* function of ik. Hence only odd derivatives of ω with respect to k exist when $k \to 0$. Although we wish once more to stress the prime importance of the general case, we shall now consider briefly the application of our ideas to long-wave theory.

We shall consider two cases of long waves: in the first case we shall suppose that there is dispersion only and in the second case that there is both dispersion and dissipation. So in the first case we have a wave which is purely dispersive, thus in (21) we may put $\Omega_{0i} = 0$, and α and γ will both be real. (Note, however, that γ will be of the same order as the wavenumber k_0.) A problem of this kind has already been examined by Asano, Taniuti & Yajima (1969), who obtained (21) with $\Omega_{0i} = 0$ by the method of multiple scales. They used the transformation

$$\psi = \rho^{\frac{1}{2}} \exp\left\{i\int\frac{\sigma dx}{\gamma}\right\} \tag{28}$$

in order to write the real and imaginary parts of (21) as

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho\sigma) = 0 \tag{29}$$

and

$$\frac{\partial \sigma}{\partial t}+\sigma\frac{\partial \sigma}{\partial x}+\alpha\gamma\frac{\partial \rho}{\partial x} = \frac{\gamma^2}{4}\frac{\partial}{\partial x}\left[\rho^{-\frac{1}{2}}\frac{\partial}{\partial x}\left\{\rho^{-\frac{1}{2}}\frac{\partial \rho}{\partial x}\right\}\right]. \tag{30}$$

If $\alpha\gamma$ is negative small disturbances about a constant equilibrium state ρ_0, σ_0 are unstable, though the wave may be trapped by the nonlinear potential. This is because γ is of the same order as k_0, and when $k_0, \gamma \to 0$ the system described by (29) and (30) is elliptic. However if $\alpha\gamma$ is positive a far-field analysis indicates that the departures of ρ and σ from their equilibrium values satisfy the Korteweg–de Vries equation. An additional point, which is of deeper significance, is that the two steady-state solutions of the Korteweg–de Vries equation (the solitary wave and the cnoidal wave) are both exact solutions of (29) and (30), as Tsuzuki (1971) has demonstrated.

We can understand this connexion with the Korteweg–de Vries equation from a different and interesting point of view. For long dispersive waves the linear dispersion relation may be expanded as a Taylor series for small values of k: $\Omega(k) = ak+bk^3+\ldots$, where a, b, ... are real, usually non-zero, constants. So we have a special case for which $\partial^2\Omega/\partial k^2$ vanishes as $k \to 0$ but $\partial^3\Omega/\partial k^3 \to b \neq 0$. Thus, rather than using (21) in the limit as $k, \gamma \to 0$, it is preferable and more relevant to consider the Schrödinger equation (27) with $m = 3$† and $\Omega_{0t} = 0$, so that

$$i\frac{\partial \psi}{\partial t}-\frac{i\gamma_3}{6}\frac{\partial^3 \psi}{\partial x^3} = \alpha|\psi|^2\psi. \tag{31}$$

However, (31) is not quite correct because we have the special case when $k_0 = 0$, for which α will be of order k as $k \to 0$ since ω is of order k. Now in an Eulerian formulation the nonlinear term is of the form $(\mathbf{u}.\nabla)\mathbf{u}$ or similar to uu_x, where u is a typical velocity component. For long waves the nonlinear term $|\psi|^2\psi$ in (21) and (27) comes essentially from expressing ψ as a Fourier series in e^{ikx} and extracting the coefficient of e^{ikx} from a dominant term $\psi\psi_x$ after calculating nonlinear interactions such as $ik\psi\bar{\psi}.\psi e^{ikx}$ and $-ik\psi^2 e^{2ikx}.\bar{\psi} e^{-ikx}$, where $\bar{\psi}$ is the complex conjugate of ψ. Therefore, in the long-wave limit $k \to 0$,

$$\psi\psi_x \sim ik|\psi|^2\psi,$$

that is, the nonlinear term may be expressed in its Eulerian form. So in (31) we may replace $ik|\psi|^2\psi$ by a term proportional to $\psi\psi_x$ and obtain

$$\frac{\partial \psi}{\partial t}-\frac{\gamma_3}{6}\frac{\partial^3 \psi}{\partial x^3} = -\alpha'\psi\frac{\partial \psi}{\partial x}, \tag{32}$$

where α' is proportional to the well-defined limit of α/k as $k \to 0$. Equation (32) is of the required Korteweg–de Vries form and in the hyperbolic case α' will be real, so we may seek a real solution for ψ.

† For some magneto-acoustic waves $b = 0$, then $m = 5$ is appropriate.

The above reasoning is rather tortuous but we cannot expect either of the Schrödinger equations (21) and (27) to be very useful for long waves because of the special way in which Whitham's theory requires the nonlinearity of the problem to be treated. It is, however, reassuring to find that the propagation of a small modulation on a wave of almost constant amplitude is governed by the Korteweg–de Vries equation when there is no dissipation because this result is in complete accord with invariant far-field theory applied to the Navier–Stokes equations for a non-dissipative problem.

In the second case we suppose that there is some dissipation and put $\gamma = \gamma_r + i\gamma_i$ and $\alpha = \alpha_r + i\alpha_i$ in (21), with Ω_{0i} non-zero and γ_i negative. To reduce the problem to Burgers' equation we use the transformation

$$\psi = \rho^{\frac{1}{2}} \exp\left\{i\int \sigma\, dx\right\}, \tag{33}$$

to write the real and imaginary parts of (21) as

$$\frac{\partial \sigma}{\partial t} + \tfrac{1}{2}\gamma_r \frac{\partial}{\partial x}\left\{\sigma^2 - \frac{1}{r}\frac{\partial^2 r}{\partial x^2}\right\} + \tfrac{1}{2}\gamma_i \frac{\partial}{\partial x}\left\{\frac{1}{\rho}\frac{\partial}{\partial x}(\rho\sigma)\right\} = -\alpha_r \frac{\partial \rho}{\partial x}, \tag{34}$$

and

$$\frac{\partial \rho}{\partial t} + \gamma_i\left\{r\frac{\partial^2 r}{\partial x^2} - \rho\sigma^2\right\} + \gamma_r \frac{\partial}{\partial x}(\rho\sigma) = 2\Omega_{0i}\rho + 2\alpha_i \rho^2, \tag{35}$$

where $r = \rho^{\frac{1}{2}}$. We now consider small perturbations about the equilibrium state $\sigma = \sigma_0 = 0$, $\rho = \rho_0 = -\Omega_{0i}/\alpha_i$, so that α_i must be negative if Ω_{0i} is positive. The appropriate far-field scaling is

$$\left.\begin{aligned} \rho &= \rho_0 + \delta\rho_1 + \ldots, \\ \sigma &= \delta^{\frac{1}{2}}\sigma_1 + \ldots, \\ \eta &= \delta^{\frac{1}{2}}x, \\ \tau &= \delta t, \end{aligned}\right\} \tag{36}$$

where the small parameter δ is a measure of the amplitude of the perturbation. If we substitute (36) in (34) and (35), then from the lowest order terms we obtain

$$\frac{\partial \sigma_1}{\partial \tau} + \gamma_r \sigma_1 \frac{\partial \sigma_1}{\partial \eta} + \tfrac{1}{2}\gamma_i \frac{\partial^2 \sigma_1}{\partial \eta^2} = -\alpha_r \frac{\partial \rho_1}{\partial \eta} \tag{37}$$

and

$$\gamma_r \frac{\partial \sigma_1}{\partial \eta} - \gamma_i \sigma_1^2 = 2\alpha_i \rho_1. \tag{38}$$

We now substitute for ρ_1 from (38) in (37) and find that σ_1 satisfies the Burgers equation

$$\frac{\partial \sigma_1}{\partial \tau} + \left(\gamma_r - \frac{\alpha_r\gamma_i}{\alpha_i}\right)\sigma_1 \frac{\partial \sigma_1}{\partial \eta} = -\tfrac{1}{2}\left(\gamma_i + \frac{\alpha_r\gamma_r}{\alpha_i}\right)\frac{\partial^2 \sigma_1}{\partial \eta^2}, \tag{39}$$

ρ_1 may then be found from (38).

In fact this perturbation solution is an exact solution of (34) and (35). In particular, σ_1 may have the steady-state tanh-profile solution of the Burgers equation, in which case ρ_1 will be a constant plus a term proportional to σ_1^2. This profile is the third *exact* steady-state solution of the Schrödinger equation. As in the first dispersive case both α_r and α_i will be of order k as $k \to 0$, so the limiting value of α_r/α_i in (39) is well-defined. If both γ_r and α_r are zero then (39) is the simple diffusion equation. In this case there exist solutions of (34) and (35) with $\sigma = 0$, and the form of $\rho^{\frac{1}{2}}$ may yield either of the two other steady-state solutions of the Schrödinger equation; however, these are both unstable when γ_i is negative. Equation (39) is invalid if $\alpha_i = 0$ for then ρ_0 does not exist since the amplitude does not directly affect the rate of dissipation; this case is more akin to the dispersive problem.

However, again, as in the first case, we can appreciate the above connexion with the Burgers equation simply by arguing that for long waves a nonlinear term $ik|\psi|^2\psi$ such as that in (21) may be replaced by a term proportional to $\psi\psi_x$. For example when $\gamma_r = 0$ we may write (21) as a slightly modified form of Burgers' equation:

$$\frac{\partial \psi}{\partial t} + \tfrac{1}{2}\gamma_i \frac{\partial^2 \psi}{\partial x^2} = \Omega_{0i}\psi - \alpha' \psi \frac{\partial \psi}{\partial x}, \tag{40}$$

where α' is proportional to the limit of α/k as $k \to 0$. If α' is real then we may seek a solution for ψ which is real, and, more generally, we can do this when $\gamma_r \neq 0$ provided that α'/γ is imaginary.

What we have shown above is that when there is dissipation the propagation of a small modulation on a wave of otherwise constant amplitude is governed by the Burgers equation. Again this is just the result which we hoped to obtain because the invariant far-field equation which governs the propagation of a small disturbance about a constant equilibrium state solution of the Navier–Stokes equations in the direction of propagation is the Burgers equation.

4. Discussion

In this paper we have used the Whitham theory of slowly varying wave trains, modified slightly to include dissipation, to investigate the propagation of a weak nonlinear wave in a fluid which is both dispersive and dissipative. If the energy of the motion is concentrated in a narrow wavenumber band we find that, in a frame of reference which moves downstream with the group velocity, the modulation may be described by a wave function ψ which satisfies a Schrödinger equation of the form

$$i\frac{\partial \psi}{\partial t} + W\frac{\partial^2 \psi}{\partial x^2} = V\psi. \tag{41}$$

In (41) W is a complex constant whose imaginary part is negative and the potential function V is of the form $\delta_1 + \delta_2|\psi|^2$, where δ_1 and δ_2 are also complex constants. The real part of δ_1 may be taken to be zero, and if the bandwidth is measured by a small parameter ϵ then δ_1 will be of order ϵ^2 and ψ will be of order ϵ. We conjecture that for stronger waves ψ will still satisfy (41) with both W and V as complex functions of $|\psi|^2$ such that if they are expanded as power series in $|\psi|^2$ then, for consistency, the expansion for W will contain one term less than that for V.

This equation is of the same form as that found by Stewartson & Stuart (1971) for describing the growth of a small disturbance in plane Poiseuille flow when the

Reynolds number is just above the critical value. More recently, Asano (1972) has also obtained (41), by use of the method of multiple scales, as the far-field equation for a hyperbolic system of partial differential equations with first-order time derivatives and second-order space derivatives.

It is significant that the relevant equation is of Schrödinger type because we have applied Whitham's theory to the propagation of a group of waves of small wavenumber differences – a *pulse*. As Schrödinger (1926) pointed out in his paper on continuous transition from micro-mechanics to macro-mechanics, this problem is just the continuum analogue of that of the wave motion of a particle in quantum mechanics. Indeed the diagram used by Schrödinger to explain the relevance of his wave equation to the propagation of a pulse is almost identical to the diagrams of wave-pulse experiments described by Feir (1967).

When there is no dissipation both W and V in (41) will be real quantities; as an example of such a problem we refer to Chu & Mei (1971), who considered the evolution of Stokes waves in deep water. They used the small amplitude equations of the Whitham theory to obtain the principal equations (2.5a, b) of their paper which may be written as

$$\frac{\partial a^2}{\partial t}+\frac{\partial}{\partial x}(\phi_x a^2) = 0 \tag{42}$$

and

$$-2\frac{\partial^2\phi}{\partial x\,\partial t}+\frac{\partial}{\partial x}\left\{-\phi_x^2+\frac{a^2}{4}+\frac{a_{xx}}{16a}\right\} = 0. \tag{43}$$

In (42) and (43) we have put their $w = -2\phi_x$; here a is a small amplitude and ϕ is a small phase variation. We wish to recast these equations into their Schrödinger form and to find W and the potential function V.

To do this we integrate (43) with respect to x. The constant of integration will be a function of time only and we may set this equal to zero as ϕ is undefined to within an arbitrary additive function of t. So we can rewrite (43) as

$$\phi_t+\tfrac{1}{2}\phi_x^2-\tfrac{1}{8}a^2-a_{xx}/32a = 0. \tag{44}$$

If we now put $\psi = a\,e^{4i\phi}$ we may readily verify that (42) and (44) are equivalent to

$$i\frac{\partial\psi}{\partial t}+\frac{1}{8}\frac{\partial^2\psi}{\partial x^2} = -\tfrac{1}{2}|\psi|^2\psi, \tag{45}$$

so that $W = \frac{1}{8}$ and $V = -\frac{1}{2}|\psi|^2$.

A similar result to (45) would have been obtained by Nayfeh & Hassan (1971) in their wave problems had they not omitted to move their co-ordinate system with the group velocity, so as to be near the centre of the wave group. Their consequent omission of a modulation term similar to a_{xx}/a in (44) led them to the erroneous conclusion that the nonlinearity affects only the phase of the motion directly and not the amplitude.

We conclude with a few comments on invariant far-field theory and the mixed Korteweg–de Vries–Burgers problem. First, we observe that in deriving (5) we assumed that the coefficient of B_{xx}/B was non-zero and that we could ignore the higher derivatives of B. So the analysis of § 2 is a far-field theory and therefore it is not surprising that the differential equation (21) which governs ψ is an invariant far-field equation; by invariant we mean here that (21) is its own far-field equation. Moreover, if the coefficient of B_{xx}/B in (5) is zero, so that (5) must be modified to include a higher derivative, then the governing equation will be (27), which is also an invariant far-field equation.

Second, we notice that for small modulations of equilibrium states and for long waves the far-field equations will usually be of the Korteweg–de Vries or Burgers family. If a physical problem with both dispersion and dissipation is controlled by a mixed Korteweg–de Vries–Burgers equation, which is *not* invariant, then three time scales are involved. For such a problem would a multiple-time-scale analysis require the initial condition for the longest time scale to be a solution of the Burgers equation? For example, Baum (1971) considered an initial-value problem in the kinetic theory of gases and obtained Burgers equation as his far-field equation in the direction of propagation because (*a*) he considered a small perturbation about an equilibrium state, (*b*) his problem was dissipative and (*c*) he used an Eulerian (as opposed to Lagrangian) formulation.

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